*International Journal of Theoretical Physics, Vol. 39, No. 3, 2000*

# Finitely Generated Free Orthomodular Lattices. III†

**M. Haviar[1] and P. Konôpka[2]**

*Received December 8, 1999*

The finitely generated free algebras $F_{\mathbf{V}(\mathbf{L}_k)}(n)$ ($k \geq 2$, $n \geq 3$) in the varieties $\mathbf{V}(\mathbf{L}_k)$ of orthomodular lattices generated by the ortholattices $\mathbf{L}_k$ which are horizontal sums of one block $\mathbf{2}^3$ and $k-1$ blocks $\mathbf{2}^2$ are described as abstract algebras. This is a continuation of earlier work and indicates the complexity one must expect when describing the finitely generated free algebras in finitely generated varieties of orthomodular lattices generated by ortholattices containing Boolean blocks larger than $\mathbf{2}^2$.

In Haviar *et al.* (1997a, b) we completely described the finitely generated free algebras in the finitely generated varieties of modular ortholattices as abstract algebras. We gave a detailed introduction to the topic as well as the necessary background, and we refer the reader to these two papers for all concepts and facts not explained here. The basic facts about orthomodular lattices can be found in Kalmbach (1983) and Beran (1984). Transferring the known 'concrete representations' of the finitely generated free algebras (as algebras of term functions) to 'abstract representations' enables us to derive easily the cardinalities of such free algebras, a fact we used fully in the first two papers.

The finitely generated subvarieties of the variety $\mathcal{MO}$ of all modular ortholattices form the chain

$$\mathcal{T} \subsetneq \mathcal{B} \subsetneq \mathcal{MO}_2 \subsetneq \mathcal{MO}_3 \subsetneq \cdots \subsetneq \mathcal{MO}_k \subsetneq \mathcal{MO}_{k+1} \subsetneq \cdots \subsetneq \mathcal{MO}$$

of type $\omega + 1$, where $\mathcal{T}$ and $\mathcal{B}$ are the varieties of trivial algebras and

† This paper is dedicated to the memory of Prof. Gottfried T. Rüttimann.

[1] School of Mathematics, La Trobe University, Bundoora, Victoria 3083, Australia, and Department of Mathematics, PF, M. Bel University, 974 01 Banská Bystrica, Slovakia; e-mail: haviar@bb.sanet.sk.
[2] Department of Mathematics, FPV, M. Bel University, 974 01 Banská Bystrica, Slovakia; e-mail: konopka@fpv.umb.sk.

Boolean algebras, respectively, and $\mathcal{MO}_k = \mathbf{V}(\mathbf{MO}_k)$ is the variety generated by the orthomodular lattice $\mathbf{MO}_k$ of height 2 with $k$ Boolean blocks $\mathbf{2}^2$. In this paper we shall consider the chain of varieties $\mathbf{V}(\mathbf{L}_k)$ ($k \geq 2$), where $\mathbf{L}_k$ is the ortholattice which is the horizontal sum of one block $\mathbf{2}^3$ and $k - 1$ blocks $\mathbf{2}^2$. This chain is such that for every $k \geq 2$, $\mathbf{V}(\mathbf{L}_k)$ contains the variety $\mathbf{V}(\mathbf{MO}_k)$ and the variety $\mathbf{V}(\mathbf{L}_2)$ covers the variety $\mathbf{V}(\mathbf{MO}_2)$ (Kalmbach, 1983).

The aim of this paper is to describe the finitely generated free algebras $F_{\mathbf{V}(\mathbf{L}_k)}(n)$ ($k \geq 2$, $n \geq 3$) with $n$ generators in the varieties $\mathbf{V}(\mathbf{L}_k)$ as abstract algebras (see Theorem 3). We know that these free algebras are finite because the varieties $\mathbf{V}(\mathbf{L}_k) = \mathbf{ISP}(\mathbf{L}_k)$ are locally finite (Clark and Davey, 1998, Chapter 1.3).

As it was the case for the ortholattices $\mathbf{MO}_k$, the term function

$$p(x, y, z) = (x \vee z) \wedge (x \vee y') \wedge (z \vee y') \wedge [(c(x, y) \wedge z) \vee (c(y, z) \wedge x) \vee (c(x, z) \wedge x \wedge z)]$$

is an arithmeticity term function for the ortholattices $\mathbf{L}_k$, too. Indeed, if $x$, $z$ belong to the same block of $\mathbf{L}_k$, then $(x \vee z) \wedge (x' \vee z) = z$ and $c(x, z) = 1$; if $x$, $z$ are from different blocks of $\mathbf{L}_k$, then $(x \vee z) \wedge (x' \vee z) = 1$ and $c(x, z) = 0$. Thus by Theorems 2.1 and 2.2 in Haviar *et al.* (1997b), following from the Arithmetic Strong Duality Theorem in Clark and Davey (1998, Theorem 3.11), the $n$-generated free algebra $F_{\mathbf{V}(\mathbf{L}_k)}(n)$ ($k \geq 2$, $n \geq 3$) is isomorphic to the algebra of all functions from $L_k^n$ to $L_k$ preserving all partial endomorphisms of $\mathbf{L}_k$.

We proceed in a manner analogous to Haviar *et al.* (1997b). First, the $n$-generated free algebra $F_{\mathbf{V}(\mathbf{L}_k)}(n)$ can be expressed as the product

$$F_{\mathbf{V}(\mathbf{L}_k)}(n) = [0, c(x_1, \ldots, x_n)] \times [0, c'(x_1, \ldots, x_n)]$$

where $c(x_1, \ldots, x_n) = \bigwedge_{(i_1,\ldots,i_n) \in \{0,1\}^n} (x_1^{i_1} \wedge \cdots \wedge x_n^{i_n})$ is the commutator of the generators $x_1, \ldots, x_n$ of the free algebra $F_{\mathbf{V}(\mathbf{L}_k)}(n)$ [here $x_i^0 = x_i$, $x_i^1 = x_i'$ and $c'(x_1, \ldots, x_n)$ denotes $(c(x_1, \ldots, x_n))'$]. The interval $[0, c(x_1, \ldots, x_n)]$ is isomorphic to the $n$-generated free Boolean algebra $F_{\mathcal{B}}(n) \cong \mathbf{2}^{2^n}$ (Haviar *et al.*, 1997b, Theorem 3.1). Second, to evaluate $[0, c'(x_1, \ldots, x_n)]$, we decompose the interval $[0, c'(x_1, \ldots, x_n)]$ by the commutators $c(x_i, x_j)$ ($i, j = 1, \ldots, n$, $i < j$) as

$$[0, c'(x_1, \ldots, x_n)] \cong \prod_{\tilde{w} \in \{0,1\}^N} \left[0, \bigwedge_{\substack{i,j=1 \\ i<j}}^{n} c^{w_{i,j}}(x_i, x_j) \wedge c'(x_1, \ldots, x_n)\right]$$

where the product is taken over all $N$-tuples $\tilde{w} = (w_{1,2}, \ldots, w_{n-1,n}) \in \{0, 1\}^N$, $N = \binom{n}{2}$ and

$$c^{w_{i,j}}(x_i, x_j) = \begin{cases} c(x_i, x_j) & \text{if} \quad w_{i,j} = 0 \\ c'(x_i, x_j) & \text{if} \quad w_{i,j} = 1 \end{cases}$$

Here again the term function

$$t_{\tilde{w}}(x_1, \ldots, x_n) = \bigwedge_{\substack{i,j=1 \\ i<j}}^{n} c^{w_{i,j}}(x_i, x_j) \wedge c'(x_1, \ldots, x_n)$$

corresponds to a labeled unoriented graph $G_{\tilde{w}}$ (without multiple edges and loops) on the vertex set $(x_1, \ldots, x_n)$ with edges $x_i x_j$ whenever $w_{i,j} = 1$ for $i < j$. Any one of $G$, $\tilde{w}$, or $t_{\tilde{w}}(x_1, \ldots, x_n)$, the last denoted also by $C_G(x_1, \ldots, x_n)$, determines the other two. A necessary and sufficient condition on the structure of the graph $G$ for the interval $[0, t_{\tilde{w}}(x_1, \ldots, x_n)] = [0, C_G(x_1, \ldots, x_n)]$ in $F_{\mathbf{V}(\mathbf{L}_k)}(n)$ to be nontrivial can be described as in Haviar *et al.* (1997b).

*Proposition 1* (Haviar *et al.*, 1997b). The following conditions are equivalent:

(a) $C_G(x_1, \ldots, x_n)$ is not identically equal to zero.

(b) There exist elements $a_1, \ldots, a_n \in \mathbf{L}_k$ with the following properties:

- (i) $C_G(a_1, \ldots, a_n) = 1$.
- (ii) The elements $a_1, \ldots, a_n$ are not all from the same block of $\mathbf{L}_k$.
- (iii) $x_i x_j$ is an edge of $G$ if and only if $a_i$, $a_j$ are elements of different blocks in $\mathbf{L}_k$.

(c) $G_p := G$ consists of $l$ isolated vertices $(0 \le l \le n - p)$ and one connected component which is a complete $p$-partite graph $(2 \le p \le k)$.

The interval $[0, C_G(x_1, \ldots, x_n)]$ is isomorphic to the algebra of all functions from $L_k^n$ to $L_k$ which are pointwise less than or equal to $C_G(x_1, \ldots, x_n)$ and preserve all partial endomorphisms of $\mathbf{L}_k$. Any such function must take value zero whenever the term $C_G$ does. Let $T_G$ be the set of all $n$-tuples $\underline{\mathbf{a}} = (a_1, \ldots, a_n)$ from $(L_k)^n$ at which $C_G$ is nonzero, that is $C_G(a_1, \ldots, a_n) = 1$. Let us call the coordinates $a_i \in \{0, 1\}$ corresponding to isolated vertices of $G$ *trivial*. Proposition 1 yields that the nontrivial coordinates of $\underline{\mathbf{a}} \in T_G$ lie in exactly $p$ of the $k$ Boolean blocks $B_1, \ldots, B_k$ of $\mathbf{L}_k$ corresponding to the blocks of the $p$-partite component of the graph $G = G_p$, $2 \le p \le k$. Let us assume that the blocks of the $p$-partite component of the graph $G$ have cardinalities $k_1, \ldots, k_p$, where $k_1 \ge k_2 \ge \cdots \ge k_p \ge 1$ and $\Sigma_{i=1}^{p} k_i \le n$. W.l.o.g., let $(B_1, \ldots, B_p)(\underline{\mathbf{a}})$ be a sequence of the $p$ Boolean blocks of $\mathbf{L}_k$ containing the nontrivial coordinates of $\underline{\mathbf{a}}$ such that the number of the nontrivial coordinates of $\underline{\mathbf{a}}$ from the block $B_i$ is $k_i$, $i = 1, \ldots, p$. In the first

step we shall consider a partition of $T_G$ into orbits under the action of the automorphism group $\mathrm{Aut}(\mathbf{L}_k)$ and we shall count the number of orbits of $\mathrm{Aut}(\mathbf{L}_k)$ on $T_G$.
We shall distinguish types I and II of the $n$-tuples $\underline{\mathbf{a}} = (a_1, \ldots, a_n) \in T_G$ [and the corresponding orbits $\mathrm{Orb}(a)$] in case the block $\mathbf{2}^3$ of $\mathbf{L}_k$ is a member of $(B_1, \ldots, B_p)(\underline{\mathbf{a}})$, i.e., $B_i \cong \mathbf{2}^3$ for a unique $i \in \{1, \ldots, p\}$. By type I we shall mean the $n$-tuples $\underline{\mathbf{a}}$ [orbits $\mathrm{Orb}(\underline{\mathbf{a}})$] such that the $k_i$ coordinates of $\underline{\mathbf{a}}$ belonging to the block $B_i = \{0, b, b', c, c', d, d', 1\}$ are only from the set $\{b, b'\}$ for an atom $b$ of $B_i$, and by type II we shall mean the $n$-tuples $\underline{\mathbf{a}}$ [orbits $\mathrm{Orb}(\underline{\mathbf{a}})$] such that the $k_i$ nontrivial coordinates of $\underline{\mathbf{a}}$ belonging to the block $B_i$ contain distinct elements $b$, $c$, where $b$, $c$ are not an atom and its complement in $B_i$. For simplicity, let us now assume that $i = 1$ and the first $k_1$ coordinates of $\underline{\mathbf{a}}$ are from $B_1 \cong \mathbf{2}^3$; in this case, let the considered types of the $n$-tuples $\underline{\mathbf{a}}$ [orbits $\mathrm{Orb}(\underline{\mathbf{a}})$] be specified as I.1 and II.1, respectively. Since there are automorphisms of $\mathbf{L}_k$ permuting any two of the three atoms $b$, $c$, $d$ of the block $B_1$ and permuting the atoms $a_j$, $a_j'$ of other blocks $B_2$, $\ldots$, $B_p$, to pick up a representative of an orbit $\mathrm{Orb}(\underline{\mathbf{a}})$of type I.1, we have $2^{k_1}$ choices for the $k_1$ coordinates from $B_1$, $2^{k_i-1}$ choices for the $k_i$ coordinates from $B_i$ ($i \in \{2, \ldots, p\}$), and $2^{n-(k_1+\cdots+k_p)}$ choices for the coordinates of $\underline{\mathbf{a}}$ from $\{0, 1\}$. This altogether gives

$$2^{k_1} \cdot 2^{k_2-1} \cdot \cdots \cdot 2^{k_p-1} \cdot 2^{n-(k_1+\cdots+k_p)} = 2^{n-p+1}$$

different orbits $\mathrm{Orb}(\underline{\mathbf{a}})$ of $\mathrm{Aut}(\mathbf{L}_k)$ of type I.1 on $T_G$. [We shall later show that among all orbits $\mathrm{Orb}(\underline{\mathbf{a}})$ of type I it is sufficient to consider only the orbits of type I.1.] The number of orbits $\mathrm{Orb}(\underline{\mathbf{a}})$ of type II.1 in $T_G$ under the automorphism action will follow from the following lemma.

*Lemma 2.* There are (up to the automorphism action)

$$P(k) = 2^{k-1} + 6^{k-1}$$

choices for the $k := k_1$ coordinates of the $n$-tuples $\underline{\mathbf{a}} = (a_1, \ldots, a_n)$ of type I.1 or II.1 in $T_G$ to be taken from the block $B_1 \cong \mathbf{2}^3$.

*Proof.* If the pair of the first two coordinates of $\underline{\mathbf{a}}$ taken from $B_1$ is one of the four pairs $(b, c)$, $(b, c')$, $(b', c)$, $(b', c')$, where the distinct elements $b, c \notin \{0, 1\}$ are not an atom of $B_1$ and its complement, then any of the remaining $k - 2$ coordinates from $B_1$ can be chosen arbitrarily from the six elements $\{b, b', c, c', d, d'\}$ of $B_1$, giving $4 \cdot 6^{k-2}$ choices for the $k$ coordinates from the block $B_1$ starting with such first two coordinates. In the other case the pair of the first two coordinates is one of $(b, b)$, $(b, b')$, $(b', b)$, $(b', b')$ for an atom $b$ of $B_1$, giving (up to the automorphism action) two choices $b$ and $b'$ for the first coordinate and, recursively, $P(k - 1)$ choices for the remaining $k - 1$ coordinates. Hence we arrive at the recursive formula

$$P(k) = 4 \cdot 6^{k-2} + 2 \cdot P(k-1)$$

By standard methods of solving such formulas, we obtain

$$\frac{P(k) - 2P(k-1)}{P(k-1) - 2P(k-2)} = 6$$

$$P(k) - 8P(k-1) + 12P(k-2) = 0$$

$$u^2 - 8u + 12 = 0$$

$$u_1 = 2, \qquad u_2 = 6$$

$$P(k) = \alpha \cdot 2^k + \beta \cdot 6^k, \qquad \alpha, \beta \in R$$

One can check that $P(2) = 8$ and $P(3) = 40$, which leads to

$$\alpha = \frac{1}{2}, \qquad \beta = \frac{1}{6}$$

Hence $P(k) = 2^{k-1} + 6^{k-1}$. ■

Since again there are automorphisms permuting the atoms $a_j$, $a_j'$ of other blocks $B_2, \ldots, B_p$, to pick up a representative $\underline{\mathbf{a}}$ of an orbit Orb($\underline{\mathbf{a}}$) of one of the types I.1, II.1, we have $2^{k_1-1} + 6^{k_1-1}$ choices for the coordinates from $B_1$, $2^{k_i-1}$ choices for the coordinates from $B_i$ for $i = 2, \ldots, p$, and $2^{n-(k_1+\cdots+k_p)}$ choices for the coordinates of $\underline{\mathbf{a}}$ from $\{0, 1\}$. This altogether gives

$$(2^{k_1-1} + 6^{k_1-1}) \cdot 2^{k_2-1} \cdot \cdots \cdot 2^{k_p-1} \cdot 2^{n-(k_1+\cdots+k_p)} = 2^{n-p}(3^{k_1-1} + 1)$$

orbits Orb($\underline{\mathbf{a}}$) of Aut($\mathbf{L}_k$) of type I.1 or II.1 on $T_G$. Hence the number of orbits Orb($\underline{\mathbf{a}}$) of type II.1 is $2^{n-p}(3^{k_1-1} + 1) - 2^{n-p+1} = 2^{n-p}(3^{k_1-1} - 1)$ and the number of orbits Orb($\underline{\mathbf{a}}$) of type II altogether is

$$N(k_1, \ldots, k_p) = 2^{n-p}\left[\left(\sum_{i=1}^{p} 3^{k_i-1}\right) - p\right]$$

Let us assume now that for the $n$-tuple $\underline{\mathbf{a}} = (a_1, \ldots, a_n) \in T_G$, the block $\mathbf{2}^3$ of $\mathbf{L}_k$ is not a member of $(B_1, \ldots, B_p)(\underline{\mathbf{a}})$, hence $B_i \cong \mathbf{2}^2$ for all $i = 1, \ldots, p$. Let us say such $n$-tuples $\underline{\mathbf{a}} \in T_G$ [the corresponding orbits Orb($\underline{\mathbf{a}}$)] are of type III. Because there are automorphisms permuting the atoms $a_j$, $a_j'$ of any of the blocks $B_1, \ldots, B_p$, there are obviously

$$2^{k_1-1} \cdot 2^{k_2-1} \cdot \cdots \cdot 2^{k_p-1} \cdot 2^{n-(k_1+\cdots+k_p)} = 2^{n-p}$$

orbits Orb($\underline{\mathbf{a}}$) of Aut($\mathbf{L}_k$) of type III.

In the second step we determine the structure of the Aut($\mathbf{L}_k$)-preserving functions from $L_k^n$ to $L_k$ which are pointwise less than or equal to $C_G(x_1, \ldots, x_n)$. We proceed as in Haviar *et al.* (1997b). We may extend the action of

$\mathrm{Aut}(\mathbf{L}_k)$ on $\mathbf{L}_k$ pointwise to $(\mathbf{L}_k)^n$, so that for $\underline{\mathbf{a}} = (a_1, \ldots, a_n) \in (\mathbf{L}_k)^n$ and $\alpha \in \mathrm{Aut}(\mathbf{L}_k)$, $\underline{\mathbf{a}}^\alpha = (a_1^\alpha, \ldots, a_n^\alpha) \in (\mathbf{L}_k)^n$ and a function $f: (\mathbf{L}_k)^n \to \mathbf{L}_k$ is $\alpha$-preserving if for all $\underline{\mathbf{a}} \in (\mathbf{L}_k)^n$, $f(\underline{\mathbf{a}}^\alpha) = f(\underline{\mathbf{a}})^\alpha$. To define an $\mathrm{Aut}(\mathbf{L}_k)$-preserving function $f \leq C_G$, we cannot freely choose images from $\mathbf{L}_k$ for representatives of the orbits $\mathrm{Orb}(\underline{\mathbf{a}})$ within $T_G$ because when $p < k$, there exist automorphisms $\alpha \neq \beta$ in $\mathrm{Aut}(\mathbf{L}_k)$ such that for any representative $\underline{\mathbf{a}}$ of orbit $\mathrm{Orb}(\underline{\mathbf{a}})$, $\underline{\mathbf{a}}^\alpha$ is equal to $\underline{\mathbf{a}}^\beta$, restricting the choices for $f(\underline{\mathbf{a}})$ to those which satisfy $f(\underline{\mathbf{a}})^\alpha = f(\underline{\mathbf{a}})^\beta$. Hence, as in Haviar *et al.* (1997b), we may freely choose the image $f(\underline{\mathbf{a}})$ for each orbit-representative $\underline{\mathbf{a}}$ within $\bigcap_{\gamma \in \mathrm{Stab}_{\underline{\mathbf{a}}}} \mathrm{fix}_{\mathbf{L}_k}(\gamma)$, which forces the values of the other elements $\underline{\mathbf{a}}^\alpha$ in $\mathrm{Orb}(\underline{\mathbf{a}})$ to be $f(\underline{\mathbf{a}}^\alpha) = f(\underline{\mathbf{a}})^\alpha$. The difference from Haviar *et al.* (1997b) is that obviously now only the orbits of types I and III within $T_G$ contribute a factor $\mathbf{MO}_p$ to the algebra of $\mathrm{Aut}(\mathbf{L}_k)$-preserving functions $f: (\mathbf{L}_k)^n \to \mathbf{L}_k$, while the orbits of type II within $T_G$ contribute a factor $\mathbf{L}_p$. Indeed, this follows from the fact that the stabilizer of $n$-tuples $\underline{\mathbf{a}} \in T_G$ of type II with associated sequences of blocks $(B_1, \ldots, B_p)(\underline{\mathbf{a}})$, where $B_i \cong \mathbf{2}^3$ for a unique $i \in \{1, \ldots, p\}$, consists of exactly those automorphisms in $\mathrm{Aut}(\mathbf{L}_k)$ which fix all elements of the blocks $B_1, \ldots, B_p$ in $\mathbf{L}_k$ and permute only atoms in the remaining $k - p$ blocks $\mathbf{2}^2$ of $\mathbf{L}_k$.

In the third step we determine which of the orbits $\mathrm{Orb}(\underline{\mathbf{a}})$ of types I, II and III can be "glued together" by the action of the partial endomorphisms of $\mathbf{L}_k$. First, we note that for $\underline{\mathbf{a}} = (a_1, \ldots, a_n)$, $\underline{\mathbf{b}} = (b_1, \ldots, b_n)$ in $T_G$, the action $e(a_1) = b_1, \ldots, e(a_n) = b_n$ by a partial endomorphism $e$ of $\mathbf{L}_k$ is impossible if the domain $\mathrm{dom}(e)$ is just a subalgebra of one of the blocks of $\mathbf{L}_k$ because the nontrivial coordinates of $\underline{\mathbf{a}}$, $\underline{\mathbf{b}}$ from $T_G$ always lie in at least two different blocks of $\mathbf{L}_k$. Second, we note that for any partial endomorphism $e$ of $\mathbf{L}_k$ having an action $e(a_1) = b_1, \ldots, e(a_n) = b_n$ for some $(a_1, \ldots, a_n)$, $(b_1, \ldots, b_n) \in T_G$, there always exists a partial endomorphism $e'$ of $\mathbf{L}_k$ with a "reverse action" $e'(b_1) = a_1, \ldots, e'(b_n) = a_n$. Hence the binary relation $E$ on the set of orbits $\mathrm{Orb}(\underline{\mathbf{a}})$ of types I, II and III defined by $(\mathrm{Orb}(\underline{\mathbf{a}}), \mathrm{Orb}(\underline{\mathbf{b}})) \in E$ if there is a partial endomorphism $e$ with the action $e(a_1) = b_1, \ldots, e(a_n) = b_n$ is an equivalence relation.

In the first step we dealt with the $n$-tuples $\underline{\mathbf{a}} \in T_G$ of type I.1 determined by a sequence of blocks $(B_1, B_2 \ldots, B_p)(\underline{\mathbf{a}})$, where $B_1 \cong \mathbf{2}^3$. Now we see that for an arbitrary orbit $\mathrm{Orb}(\underline{\mathbf{b}})$ of type I with an associated sequence $(B_1', B_2' \ldots, B_p')(\underline{\mathbf{b}})$, where $B_i' \cong \mathbf{2}^3$ for a unique $i \in \{2, \ldots, p\}$, we obtain $(\mathrm{Orb}(\underline{\mathbf{a}}), \mathrm{Orb}(\underline{\mathbf{b}})) \in E$ for the $n$-tuple $\underline{\mathbf{a}} = (a_1, \ldots, a_n) \in T_G$ of type I.1 obtained from $\underline{\mathbf{b}}$ by mutually replacing in $\underline{\mathbf{b}}$ the coordinates from the first block $B_1' \cong \mathbf{2}^2$ with the coordinates from the $i$th block $B_i' \cong \mathbf{2}^3$. This is witnessed by the action $e(a_1) = b_1, \ldots, e(a_n) = b_n$, where the partial endomorphism $e$ of $\mathbf{L}_k$ acts such that it mutually replaces the atoms of $\mathbf{L}_k$ which are the coordinates of $\underline{\mathbf{b}}$ coming from the block $B_1' \cong \mathbf{2}^2$ with the elements $b$, $b'$ of $\mathbf{L}_k$ which are the coordinates of $\underline{\mathbf{b}}$ coming from the block $B'_i \cong \mathbf{2}^3$ and fixes all elements of the blocks $B'_2, \ldots, B'_{i-1}, B'_{i+1}, \ldots, B'_p$ of $\mathbf{L}_k$. This means that a function $f: L_k^n \to L_k$ preserving all partial endomorphisms of $\mathbf{L}_k$ can map the representative $\underline{\mathbf{a}}$ of Orb($\underline{\mathbf{a}}$) of type I.1 arbitrarily into the subalgebra $\mathbf{MO}_p$ of $\mathbf{L}_k$, and the image of the representative $\underline{\mathbf{b}}$ of Orb($\underline{\mathbf{b}}$) of a given type I (different from I.1) is determined by

$$f(b_1, \ldots, b_n) = f(e(a_1), \ldots, e(a_n)) = e(f(a_1, \ldots, a_n))$$

Consequently, the factors $\mathbf{MO}_p$ contributed by the orbits Orb($\underline{\mathbf{a}}$) of type I different from I.1 will not be considered.

Similarly, each orbit Orb($\underline{\mathbf{a}}$) where $\underline{\mathbf{a}} = (a_1, \ldots, a_n)$ is of type III with an associated sequence $(B_1, B_2 \ldots, B_p)(\underline{\mathbf{a}})$ with all blocks $B_i \cong \mathbf{2}^2$, $i = 1, \ldots, p$, can be "glued together" by the equivalence $E$ with an orbit Orb($\underline{\mathbf{b}}$) such that $\underline{\mathbf{b}} = (b_1, \ldots, b_n)$ is of type I.1 with the associated sequence $(B'_1, B_2 \ldots, B_p)(\underline{\mathbf{b}})$ where $B'_1 = \mathbf{2}^3$. This is witnessed by the action $e(a_1) = b_1, \ldots, e(a_n) = b_n$ of the partial endomorphism $e$ acting such that it mutually replaces the atoms of $\mathbf{L}_k$ which are the coordinates of $\underline{\mathbf{a}}$ coming from the block $B_1 \cong \mathbf{2}^2$ with the elements $b$, $b'$ of $\mathbf{L}_k$ which are the coordinates of $\underline{\mathbf{b}}$ coming from the block $B'_1 \cong \mathbf{2}^3$ and fixes all elements of the blocks $B_2 \ldots, B_p$ of $\mathbf{L}_k$. Consequently, the factors $\mathbf{MO}_p$ contributed by the orbits Orb($\underline{\mathbf{a}}$) of type III will not be considered either.

Finally, each orbit Orb($\underline{\mathbf{a}}$) of type I.1 with a sequence $(B_1, B_2 \ldots, B_p)(\underline{\mathbf{a}})$ such that the $k_1$ coordinates of $\underline{\mathbf{a}}$ are coming from the set $\{b, b'\}$ of the block $B_1 \cong \mathbf{2}^3$ for some atom $b$ of $B_1$, can be "glued together" by the equivalence $E$ with an obit Orb($\underline{\mathbf{b}}$) where $\underline{\mathbf{b}} = (b_1, \ldots, b_n)$ can be obtained from $\underline{\mathbf{a}} = (a_1, \ldots, a_n)$ by only mutually replacing the atom $b$ with its complement $b'$. This is clearly witnessed by the partial endomorphism of $\mathbf{L}_k$ mutually replacing $b$ and $b'$ and fixing all elements of the blocks $B_2 \ldots, B_p$ of $\mathbf{L}_k$. It will finally reduce the number of factors $\mathbf{MO}_p$ contributed by the orbits Orb($\underline{\mathbf{a}}$) of type I.1 by half, that is, $2^{n-p}$.

Hence the structure of the interval $[0, C_G(x_1, \ldots, x_n)]$ associated to a $p$-partite graph $G = G_p(k_1, \ldots, k_p)$ with blocks of cardinalitites $k_1, \ldots, k_p$ such that each $k_i \geq 1$ and $\sum_{i=1}^{p} k_i \leq n$ is

$$[0, C_G(x_1, \ldots, x_n)] \cong (\mathbf{MO}_p)^{2^{n-p}} \times (L_p)^{N((k_1, \ldots, k_p)}$$

where $N(k_1, \ldots, k_p) = 2^{n-p}[(\sum_{i=1}^{n} 3^{k_i - 1}) - p]$. We see that unlike the situation in Haviar *et al.* (1997b), this structure now depends on the sequence $(k_1, \ldots, k_p)$ of the cardinalities of the blocks of the $p$-partite graph $G$ where we can assume that $k_1 \geq \cdots \geq k_p$.

The number $\phi(n; k_1, \ldots, k_p)$ of the $p$-partite graphs $G = G_p(k_1, \ldots, k_p)$ on an $n$-element vertex set with blocks of cardinalities $k_1, \ldots, k_p$ ($k_1 \geq \cdots \geq k_p \geq 1$ $\sum_{i=1}^{p} k_i \leq n$) and with $l = n - \sum_{i=1}^{p} k_i$ isolated vertices can be determined as follows. First, we have $\binom{n}{l}$) choices for the isolated vertices. Second, the number of partitions of a labeled $(n - l)$-element set $S = \{1, \ldots, n - l\}$ into exactly $p$ blocks $S^1, \ldots, S^p$ of cardinalities $k_1, \ldots, k_p$, respectively is given by (Aigner 1979, 3.15)

$$\begin{aligned} S(n - l; k_1, \ldots, k_p) &= P(b_1, \ldots, b_{n-l}) \\ &= \frac{(n - l)!}{b_1! b_2! \ldots b_{n-l}! (2!)^{b_2} \ldots ((n - l)!)^{b_{n-l}}} \end{aligned} \tag{1}$$

where for $i = 1, \ldots, n - l$, $b_i$ denotes the number of blocks of cardinality $i$ among the blocks $S^1, \ldots, S^p$. Hence we obtain

$$\phi(n; k_1, \ldots, k_p) = \binom{n}{\sum_{i=1}^p k_i} S\left(\sum_{i=1}^p k_i; k_1, \ldots, k_p\right) \tag{2}$$

We consequently arrive to the following theorem.

*Theorem 3.* For any $n \geq 3$, $k \geq 2$, the finitely generated free algebra $F_{\mathbf{V}(\mathbf{L}_k)}(n)$ is isomorphic to the product

$$2^{2^n} \times \prod_{p=2}^{k} \prod_{\substack{(k_1,\ldots,k_p) \\ k_1 \geq \cdots \geq k_p \geq 1 \\ \sum_{i=1}^p k_i \leq n}} [(\mathbf{MO}_p)^{2^{n-p}} \times (\mathbf{L}_p)^{N(k_1,\ldots,k_2)}]^{\phi(n;k_1,\ldots,k_p)}$$

where $N(k_1, \ldots, k_p) = 2^{n-p} [(\sum_{i=1}^n 3^{k_i-1}) - p]$ and $\phi(n; k_1, \ldots, k_p)$ is given by (1) and (2). ■

One can verify that

$$\sum_{\substack{(k_1,\ldots,k_p) \\ k_1 \geq \cdots \geq k_p \geq 1 \\ \sum_{i=1}^p k_i = n-l}} S(n - l; k_1, \ldots, k_p) = S(n - l, p)$$

where the Stirling number $S(n - l, p)$ of the second kind is the number of partitions of a labeled $(n - l)$-element set into exactly $p$ parts and is given by the formula

$$S(n - l, p) = \frac{1}{p!} \sum_{s=1}^{p} (-1)^{p-s} \binom{p}{s} s^{n-1}$$

(Aigner, 1979, 3.39). From this it follows that

$$\mathbf{2}^{2^n} \times \prod_{p=2}^{k} \prod_{\substack{(k_1,\ldots,k_2)\\ k_1 \geq \cdots \geq k_p \geq 1\\ \Sigma_{i=1}^{p} k_i \leq n}} [(\mathbf{MO}_p)^{2^{n-p}}]^{\phi(n;k_1,\ldots,k_p)} = \mathbf{2}^{2^n} \times \prod_{p=2}^{k} (\mathbf{MO}_p)^{(2^{n-p}\phi'(n,p))}$$

with

$$\phi'(n, p) = \sum_{l=0}^{n-p} \binom{n}{l} S(n - l, p)$$

which is isomorphic to the $n$-generated free modular ortholattice $F_{\mathcal{MO}_k}(n)$ in the variety $\mathcal{MO}_k$ (Haviar *et al.*, 1997b). Hence we can deduce the final result.

*Corollary 4.* For any $n \geq 3$, $k \geq 2$,

$$F_{\mathbf{V(L}_k)}(n) \cong F_{\mathcal{MO}_k}(n) \times \prod_{p=2}^{k} \prod_{\substack{(k_1,\ldots,k_p)\\ k_1 \geq \cdots \geq k_p \geq 1\\ \Sigma_{i=1}^{p} k_i \leq n}} [(\mathbf{L}_p)^{N(k_1,\ldots,k_p)}]^{\phi(n;k_1,\ldots,k_p)}$$

where $F_{\mathcal{MO}_k}(n)$ is the $n$-generated free modular ortholattice in the variety $\mathcal{MO}_k$, $N(k_1, \ldots, k_p) = 2^{n-p}[(\Sigma_{i=1}^{p} 3^{k_i - 1}) - p]$, and $\phi(n; k_1, \ldots, k_p)$ is given by (1) and (2).

## ACKNOWLEDGMENTS

The authors would like to thank B. A. Davey (La Trobe University, Melbourne) for his helpful remarks. M. H. was been supported by a La Trobe University Postdoctoral Research Fellowship and partly by Slovak VEGA grant 1/4057/97.

## REFERENCES

Aigner, M. (1979). *Combinatorial Theory*, Springer-Verlag, Berlin.
Beran, L. (1984). *Orthomodular Lattices. Algebraic Approach*, Academia, Prague.
Bruns, G. (1976). Free ortholattices, *Can. J. Math.* **28**, 977–985.
Clark, D. M., and Davey, B. A. (1998). *Natural Dualities for the Working Algebraist*, Cambridge University Press, Cambridge.
Haviar, M., Konôpka, P., Priestley, H. A., and Wegener, C. B. (1997a). Finitely generated free modular ortholattices I, *Int. J. Theor. Phys.* **36**, 2639–2660.
Haviar, M., Konôpka, P., and Wegener, C. B. (1997b). Finitely generated free modular ortholattices II, *Int. J. Theor. Phys.* **36**, 2661–2679.
Kalmbach, G. (1983). *Orthomodular Lattices*, Academic Press, New York.
Pták, P., and Pulmanová, S. (1991). *Orthomodular Structures as Quantum Logics*, Kluwer, Dordrecht.